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LETTER TO THE EDITOR

Soliton solutions for a perturbed nonlinear Schrödinger equation

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Abstract. Using the inverse scattering transform we found one-parameter and the breather-like four-parameter soliton solutions of a perturbed nonlinear Schrödinger equation which describes the pulse propagation in optical fibres in the femtosecond regime.

Optical solitons in fibres are pulses which propagate without any change in pulse shape or intensity. Due to their remarkable stability properties, optical solitons are now at the centre of an active research field of nonlinear wave propagation in optical fibres [1–9]. Recently [4–8], by using an asymptotic perturbation technique it was shown that the pulse propagation in optical fibres in the femtosecond regime can be fairly described by the perturbed nonlinear Schrödinger equation (PNLSE):

$$\mathrm{i}\frac{\partial q}{\partial Z}+\frac{1}{2}\frac{\partial^2 q}{\partial T^2}+|q|^2 q+\mathrm{i}\varepsilon\left(\beta_1\frac{\partial^3 q}{\partial T^3}+\beta_2|q|^2\frac{\partial q}{\partial T}+\beta_3 q\frac{\partial |q|^2}{\partial T}\right)=0 \tag{1}$$

where q represents a normalized complex amplitude of the pulse envelope, Z is a normalized distance along the fibre, T is the normalized retarded time (we employ a frame of reference moving with the pulse with its group velocity), ε is a small parameter and $\beta_1, \beta_2, \beta_3$ are the real normalized parameters which depend on the fibre characteristics (β_1 is the coefficient of the linear higher order dispersion effect and β_2, β_3 are overlap integrals [6]). A new model, to include saturation effects of the Kerr nonlinearity, has been recently derived [10], in which the governing equation is a combination of the exponential nonlinear Schrödinger equation (NLSE) and the derivative one. For $\varepsilon=0$ in equation (1) we obtain the standard NLSE, which is one of the complete integrable nonlinear partial differential equations. Its solutions can be obtained by different methods, e.g., by using the inverse scattering transform (IST) [11–17], the Lie group theory [18], by constructing a certain completely integrable finite dimensional dynamic system whose solutions determine the exact solutions of the NLSE [19–21], etc. We mention also the recent work on IST perturbation theory for soliton propagation and the first and the second-order perturbation expansion for soliton propagation in optical fibres [22].

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To the best of our knowledge for arbitrary parameters $\beta_1, \beta_2, \beta_3$ the equation (1) is not completely integrable, but for an appropriate choice of these parameters it can be integrated by the IST. Thus the cases when $\beta_1:\beta_2:\beta_3=0:1:1$ (the derivative NLSE type I), $\beta_1:\beta_2:\beta_3=0:1:0$ (the derivative NLSE type II) and $\beta_1:\beta_2:\beta_3=1:6:0$ (the Hirota equation) were solved in [23], [24] and [25], respectively.

Recently, in [26] it was shown that the case $\beta_1:\beta_2:\beta_3=1:6:3$ is also integrable by using the IST. We mention that in [26] the authors did not obtain the most general single-soliton solution because they did not take properly into account the symmetry properties of the matrix U in the U–V representation for the equation (1).

In the present letter we find for this case the general single-soliton solution classified by the following criteria:

(i) the diagonal element of the scattering matrix $\alpha_{33}(\zeta)$ has only one zero on the imaginary axis;

(ii) the diagonal element of the scattering matrix $\alpha_{33}(\zeta)$ has two zeros located at the positions symmetric with respect to the imaginary axis.

With the specific choice of the parameters which describe the general solution we find the 'breather' single-soliton solution for (1).

In order to integrate (1) in the case $\beta_1:\beta_2:\beta_3=1:6:3$ we make, as in [26], the following transformation:

$$u(x,t)=q(T,Z)\exp\left[-\frac{\mathrm{i}}{6\varepsilon}\left(T-\frac{Z}{18\varepsilon}\right)\right] \tag{2}$$

with $t=Z$ and $x=T-\frac{Z}{12\varepsilon}$.

Thus (1) transforms to a complex modified KdV-type equation:

$$\frac{\partial u}{\partial t}+\varepsilon\left(\frac{\partial^3 u}{\partial x^3}+6|u|^2\frac{\partial u}{\partial x}+3u\frac{\partial |u|^2}{\partial x}\right)=0. \tag{3}$$

In order to integrate equation (3) by IST we consider the following eigenvalue problem:

$$\frac{\partial \Psi}{\partial x}=U\Psi \tag{4}$$

where

$$U=\begin{pmatrix} -\mathrm{i}\zeta & 0 & u \\ 0 & -\mathrm{i}\zeta & u^* \\ -u^* & -u & \mathrm{i}\zeta \end{pmatrix} \tag{5}$$

Ψ is a column vector: $(\Psi_1, \Psi_2, \Psi_3)'$ and ζ is a time independent spectral parameter. With the time evolution of the eigenvector Ψ given by:

$$\frac{\partial \Psi}{\partial t}=V\Psi \tag{6}$$

where

$$V=-4\,\mathrm{i}\varepsilon\zeta^3\begin{pmatrix}1&0&0\\0&1&0\\0&0&-1\end{pmatrix}+4\varepsilon(\zeta^2-|u|^2)\begin{pmatrix}0&0&u\\0&0&u^*\\-u^*&-u&0\end{pmatrix}$$
$$+2\,\mathrm{i}\varepsilon\zeta\begin{pmatrix}|u|^2&u^2&u_x\\u^{*2}&|u|^2&u_x^*\\u_x^*&u_x&-2|u|^2\end{pmatrix}-\varepsilon\begin{pmatrix}0&0&u_{xx}\\0&0&u_{xx}^*\\-u_{xx}^*&-u_{xx}&0\end{pmatrix}$$
$$+\varepsilon(uu_x^*-u_xu^*)\begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix}\tag{7}$$

the compatibility condition of equations (4) and (6) is equivalent to (3). We note that IST with 3×3 U–V matrix representation have been discussed in [27–29].

Next for a real eigenvalue ζ we introduce the Jost functions $\varphi^{(i)}(x;\zeta)$ and $\psi^{(i)}(x;\zeta)$, $i=1,2,3$ which satisfy the following asymptotic conditions:

$$\varphi_j^{(i)}(x;\zeta)\to\delta_{ij}\,\mathrm{e}^{\mathrm{i}\gamma_i\zeta x}\qquad x\to-\infty\tag{8}$$

$$\psi_j^{(i)}(x;\zeta)\to\delta_{ij}\,\mathrm{e}^{\mathrm{i}\gamma_i\zeta x}\qquad x\to\infty\tag{9}$$

where $\gamma_1=\gamma_2=-1$ and $\gamma_3=1$. Because for real ζ the matrix U is antihermitian ($U^+=-U$) we have:

$$\frac{\partial}{\partial x}(\Psi^{(1)+}\Psi^{(2)})=0\tag{10}$$

for any pair of solutions of (4) corresponding to the same eigenvalue ζ. Now we introduce the scattering matrix $\alpha=[\alpha_{ij}(\zeta)]_{i,j=1,2,3}$ via the following relationship:

$$\varphi^{(i)}(x;\zeta)=\sum_{j=1}^{3}\alpha_{ij}(\zeta)\psi^{(j)}(x;\zeta)\tag{11}$$

between the two bases $\{\varphi^{(i)}(x;\zeta)\}_{i=1,2,3}$ and $\{\psi^{(i)}(x;\zeta)\}_{i=1,2,3}$ in the space of solutions of the equation (4). Because the matrix α is unimodular ($\det\alpha=1$) and taking into account that the two bases $\{\varphi^{(i)}(x;\zeta)\}_{i=1,2,3}$ and $\{\psi^{(i)}(x;\zeta)\}_{i=1,2,3}$ are orthogonal we obtain that the matrix α is unitary, i.e., $\alpha^+=\alpha^{-1}$. Using this property and equation (11) one can easily find the following equations:

$$[\alpha_{22}(\zeta)\varphi^{(1)}\,\mathrm{e}^{\mathrm{i}\zeta x}-\alpha_{12}(\zeta)\varphi^{(2)}\,\mathrm{e}^{\mathrm{i}\zeta x}]/\alpha_{33}^*(\zeta)=\psi^{(1)}\,\mathrm{e}^{\mathrm{i}\zeta x}-[\alpha_{31}^*(\zeta)/\alpha_{33}^*(\zeta)]\psi^{(3)}\,\mathrm{e}^{\mathrm{i}\zeta x}\tag{12a}$$

$$[-\alpha_{21}(\zeta)\varphi^{(1)}\,\mathrm{e}^{\mathrm{i}\zeta x}+\alpha_{11}(\zeta)\varphi^{(2)}\,\mathrm{e}^{\mathrm{i}\zeta x}]/\alpha_{33}^*(\zeta)=\psi^{(2)}\,\mathrm{e}^{\mathrm{i}\zeta x}-[\alpha_{32}^*(\zeta)/\alpha_{33}^*(\zeta)]\psi^{(3)}\,\mathrm{e}^{\mathrm{i}\zeta x}\tag{12b}$$

$$\varphi^{(3)}\,\mathrm{e}^{-\mathrm{i}\zeta x}/\alpha_{33}(\zeta)$$
$$=\psi^{(3)}\,\mathrm{e}^{-\mathrm{i}\zeta x}+[\alpha_{31}(\zeta)/\alpha_{33}(\zeta)]\psi^{(1)}\,\mathrm{e}^{-\mathrm{i}\zeta x}+[\alpha_{32}(\zeta)/\alpha_{33}(\zeta)]\psi^{(2)}\,\mathrm{e}^{-\mathrm{i}\zeta x}.\tag{12c}$$

Using the same technique as in [28] one can show that $\varphi^{(1)}\,\mathrm{e}^{\mathrm{i}\zeta x}$, $\varphi^{(2)}\,\mathrm{e}^{\mathrm{i}\zeta x}$, $\psi^{(3)}\,\mathrm{e}^{-\mathrm{i}\zeta x}$, $\alpha_{11}(\zeta)$, $\alpha_{12}(\zeta)$, $\alpha_{21}(\zeta)$, $\alpha_{22}(\zeta)$ and $\alpha_{33}^*(\zeta^*)$ can be analytically continued in the upper complex half plane ($\mathrm{Im}\,\zeta>0$) and $\psi^{(1)}\,\mathrm{e}^{\mathrm{i}\zeta x}$, $\psi^{(2)}\,\mathrm{e}^{\mathrm{i}\zeta x}$, $\varphi^{(3)}\,\mathrm{e}^{-\mathrm{i}\zeta x}$, $\alpha_{11}^*(\zeta^*)$, $\alpha_{12}^*(\zeta^*)$, $\alpha_{21}^*(\zeta^*)$, $\alpha_{22}^*(\zeta^*)$ and $\alpha_{33}(\zeta)$ can be analytically continued in the lower complex half plane ($\mathrm{Im}\,\zeta<0$) if $|u|$ tends to zero sufficiently fast as $|x|\to\infty$.

From the explicit form of the matrix U we can deduce the following symmetry relations between the elements of the scattering matrix α:

$$\alpha_{11}(\zeta)=\alpha_{22}^*(-\zeta^*) \qquad \alpha_{12}(\zeta)=\alpha_{21}^*(-\zeta^*) \qquad \alpha_{33}(\zeta)=\alpha_{33}^*(-\zeta^*)$$
$$\alpha_{31}(\zeta)=\alpha_{32}^*(-\zeta^*) \qquad \alpha_{13}(\zeta)=\alpha_{23}^*(-\zeta^*). \tag{13}$$

Next we derive the Gel'fand–Levitan–Marchenko (GLM) equations. To this aim we introduce the integral representations of the Jost functions:

$$\psi_j^{(i)}(x;\zeta)=\delta_{ij}\,\mathrm{e}^{\mathrm{i}\gamma_i\zeta x}+\int_x^\infty \mathrm{d}s\,K_j^{(i)}(x,s)\,\mathrm{e}^{\mathrm{i}\gamma_i\zeta s} \tag{14}$$

where $K^{(i)}(x,s)=(K_1^{(i)}(x,s),\,K_2^{(i)}(x,s),\,K_3^{(i)}(x,s))'$ with $\lim_{s\to\infty}K^{(i)}(x,s)=0$, $i=1,2,3$.

A direct consequence of the symmetry relations (13) is the property that the zeros of $\alpha_{33}(\zeta)$ are either on the imaginary axis in the lower complex half plane or located symmetrically with respect to the imaginary axis at $(\zeta^*,-\zeta)$, $\operatorname{Im}\zeta>0$. In the following we assume that $\alpha_{33}(\zeta)$ has N pairs of simple zeros located symmetrically with respect to the imaginary axis at $(\zeta_i^*,-\zeta_i)$, $\operatorname{Im}\zeta_i>0$, $i=1,2,\ldots,N$.

For $\zeta=\zeta_i^*$ we have

$$\varphi^{(3)}(x;\zeta_i^*)=c_{31}^{(i)}\psi^{(1)}(x;\zeta_i^*)+c_{32}^{(i)}\psi^{(2)}(x;\zeta_i^*) \tag{15a}$$

and for $\zeta=-\zeta_i$, by using the symmetry relations (13) we obtain:

$$\varphi^{(3)}(x;-\zeta_i)=c_{32}^{(i)*}\psi^{(1)}(x;-\zeta_i)+c_{31}^{(i)*}\psi^{(2)}(x;-\zeta_i). \tag{15b}$$

In the same manner as in [26] we find from equations (12*a*–12*c*) the following GLM equations:

$$K^{(1)}(x,y)-\begin{pmatrix}0\\0\\1\end{pmatrix}F^*(x+y)-\int_x^\infty \mathrm{d}s\,K^{(3)}(x,s)F^*(s+y)=0 \tag{16a}$$

$$K^{(2)}(x,y)-\begin{pmatrix}0\\0\\1\end{pmatrix}F(x+y)-\int_x^\infty \mathrm{d}s\,K^{(3)}(x,s)F(s+y)=0 \tag{16b}$$

$$K^{(3)}(x,y)+\begin{pmatrix}1\\0\\0\end{pmatrix}F(x+y)+\begin{pmatrix}0\\1\\0\end{pmatrix}F^*(x+y)$$
$$+\int_x^\infty \mathrm{d}s\,K^{(1)}(x,s)F(s+y)+\int_x^\infty \mathrm{d}s\,K^{(2)}(x,s)F^*(s+y)=0 \tag{16c}$$

respectively, for $y>x$, where $F(z)$ is given by:

$$F(z)=\sum_{j=1}^N \mathrm{i}\left[\frac{c_{31}^{(j)}}{\alpha_{33}'(\zeta_j^*)}\,\mathrm{e}^{-\mathrm{i}\zeta_j^* z}+\frac{c_{32}^{(j)*}}{\alpha_{33}'(-\zeta_j)}\,\mathrm{e}^{\mathrm{i}\zeta_j z}\right]+\int_{-\infty}^\infty \frac{\mathrm{d}\zeta}{2\pi}\,\frac{\alpha_{31}(\zeta)}{\alpha_{33}(\zeta)}\,\mathrm{e}^{-\mathrm{i}\zeta z} \tag{17}$$

where prime denotes the derivative with respect to ζ.

Taking into account the integral representation for $\psi^{(3)}(x;\zeta)$ and using (4), we find the following expression for the 'potential' $u(x)$:

$$u(x)=-2K_1^{(3)}(x,x). \tag{18}$$

From equations (16*a*–16*c*) we finally obtain the GLM equation for $K_1^{(3)}(x,y)$:

$$K_1^{(3)}(x,y)+F(x+y)+\int_x^\infty \mathrm{d}z\, K_1^{(3)}(x,z) \times \int_x^\infty \mathrm{d}s[F^*(z+s)F(s+y)+F(z+s)F^*(s+y)]=0. \tag{19}$$

From the asymptotic expression of the matrix $V(|x|\to\infty)$ it is easy to determine the time-dependence of the scattering data:

$$\begin{aligned}
&\alpha_{33}(\zeta,t)=\alpha_{33}(\zeta,0) \qquad \alpha_{ij}(\zeta,t)=\alpha_{ij}(\zeta,0) \qquad i,j=1,2\\
&\alpha_{3i}(\zeta,t)=\alpha_{3i}(\zeta,0)\exp(-8\mathrm{i}\varepsilon\zeta^3 t) \qquad \alpha_{i3}(\zeta,t)=\alpha_{i3}(\zeta,0)\exp(8\mathrm{i}\varepsilon\zeta^3 t)\\
&c_{3i}^{(j)}(t)=c_{3i}^{(j)}(0)\exp(-8\mathrm{i}\varepsilon\zeta_j^{*3}t) \qquad i=1,2.
\end{aligned} \tag{20}$$

Now we can analyse the case (i) when the diagonal element $\alpha_{33}(\zeta)$ of the scattering matrix has only one zero on the imaginary axis at $\zeta^*=-\mathrm{i}\eta/2$, $\eta>0$. In addition we put $\alpha_{31}(\zeta)=0$ for real ζ.

In this case the function $F(z)$ is:

$$F(z)=a(t)\exp\left(-\frac{\eta z}{2}\right) \tag{21}$$

where

$$a(t)=\frac{\mathrm{i}c_{31}(t)}{\alpha'_{33}(-\mathrm{i}\eta/2)}.$$

Considering that the function $K_1^{(3)}(x,y)$ has the form

$$K_1^{(3)}(x,y)=K(x)\exp(-\eta y/2)$$

from equations (18)–(21) we obtain the single-soliton solution for (3):

$$u(x,t)=\frac{\eta}{\sqrt{2}}\operatorname{sech}[\eta(x-\varepsilon\eta^2 t-x_0)]\,\mathrm{e}^{\mathrm{i}\varphi_0} \tag{22}$$

where $x_0=(1/\eta)\ln(\sqrt{2}\,|a(0)|/\eta)$ and $\varphi_0=\arg a(0)$.

Thus we can write the single-soliton solution of (1) in the form:

$$q(Z,T)=\frac{\eta}{\sqrt{2}}\operatorname{sech}\left\{\eta\left[T-\left(\varepsilon\eta^2+\frac{1}{12\varepsilon}\right)Z-T_0\right]\right\}\exp\left\{\mathrm{i}\left[\frac{1}{6\varepsilon}\left(T-\frac{Z}{18\varepsilon}\right)+\varphi_0\right]\right\} \tag{23}$$

where $T_0=x_0$.

We mention that this soliton solution was also obtained in [26] in the limit $\hat{\xi}\to 0$ (see equations (38) and (51) in [26]).

Next we discuss the case (ii) when the diagonal element of the scattering matrix $\alpha_{33}(\zeta)$ has two zeros $(\zeta^*,-\zeta)$ where $\zeta=(-\xi+\mathrm{i}\eta)/2$ with $\xi,\eta\geqslant 0$.

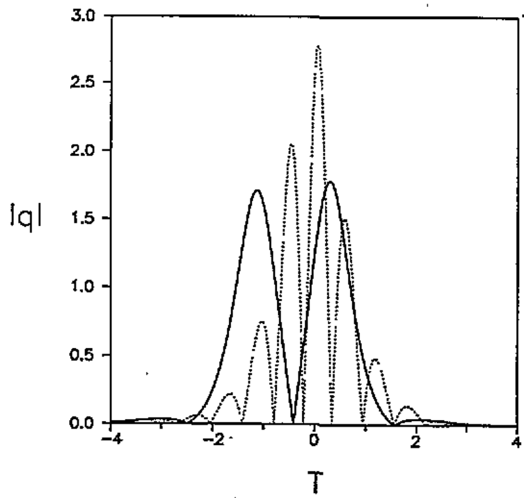

Figure 1. The shape of $|q|$ as a function of T for the breather soliton solution (27), with $\eta=2$, $\kappa=\sqrt{2}$, $\varphi_a=-\varphi_b=0$. Here $\xi=1$ (solid line) and $\xi=5$ (dotted line).

In order to find the general soliton solution in this case we consider that the function $K_1^{(3)}(x,y)$ has the following expression:

$$K_1^{(3)}(x,y)=L(x)\,\mathrm{e}^{-\mathrm{i}\zeta^* y}+M(x)\,\mathrm{e}^{\mathrm{i}\zeta y}. \tag{24}$$

As in the previous case we take $\alpha_{31}(\zeta)=0$ for real ζ so that the function $F(z)$ is:

$$F(z)=a(t)\,\mathrm{e}^{-\mathrm{i}\zeta^* z}+b(t)\,\mathrm{e}^{\mathrm{i}\zeta z} \tag{25}$$

where $a(t)=\mathrm{i}c_{31}(t)/\alpha'_{33}(\zeta^*)$ and $b(t)=\mathrm{i}c^*_{32}(t)/\alpha'_{33}(-\zeta)$.

In this case the general single-soliton solution is:

$$\begin{aligned}u(x,t)=-\frac{2\mathrm{e}^{\mathrm{i}(\varphi_a+\varphi_b)/2}}{\Delta}\,\mathrm{e}^{-A}\bigg\{&|a_0|\left[\frac{|a_0b_0|\,\mathrm{e}^{-2(A+\mathrm{i}B)}}{2\zeta^2}-\frac{(|a_0|^2+|b_0|^2)\,\mathrm{e}^{-2A}}{\eta^2}-1\right]\mathrm{e}^{\mathrm{i}B}\\&+|b_0|\left[\frac{|a_0b_0|\,\mathrm{e}^{-2(A-\mathrm{i}B)}}{\zeta^*}-\frac{(|a_0|^2+|b_0|^2)\,\mathrm{e}^{-2A}}{2\zeta}\right]\frac{\mathrm{e}^{-\mathrm{i}B}}{\mathrm{i}\eta}\\&+|b_0|\left[\frac{|a_0b_0|\,\mathrm{e}^{-2(A-\mathrm{i}B)}}{2\zeta^{*2}}-\frac{(|a_0|^2+|b_0|^2)\,\mathrm{e}^{-2A}}{\eta^2}-1\right]\mathrm{e}^{-\mathrm{i}B}\\&-|a_0|\left[\frac{|a_0b_0|\,\mathrm{e}^{-2(A+\mathrm{i}B)}}{\zeta}-\frac{(|a_0|^2+|b_0|^2)\,\mathrm{e}^{-2A}}{2\zeta^*}\right]\frac{\mathrm{e}^{\mathrm{i}B}}{\mathrm{i}\eta}\bigg\}\end{aligned} \tag{26}$$

where

$$\begin{aligned}\Delta=&\left\|\left[\frac{|a_0b_0|\,\mathrm{e}^{-2(A-\mathrm{i}B)}}{2\zeta^{*2}}-\frac{(|a_0|^2+|b_0|^2)\,\mathrm{e}^{-2A}}{\eta^2}-1\right]\right\|^2\\&-\frac{1}{\eta^2}\left\|\left[\frac{|a_0b_0|\,\mathrm{e}^{-2(A-\mathrm{i}B)}}{\zeta^*}-\frac{(|a_0|^2+|b_0|^2)\,\mathrm{e}^{-2A}}{2\zeta}\right]\right\|^2\end{aligned}$$

$A=\eta[x-\varepsilon(\eta^2-3\xi^2)t]$, $B=\xi[x+\varepsilon(\xi^2-3\eta^2)t]+(\varphi_a-\varphi_b)/2$, $a_0=a(0)$, $b_0=b(0)$, $\varphi_a=\arg a(0)$, and $\varphi_b=\arg b(0)$.

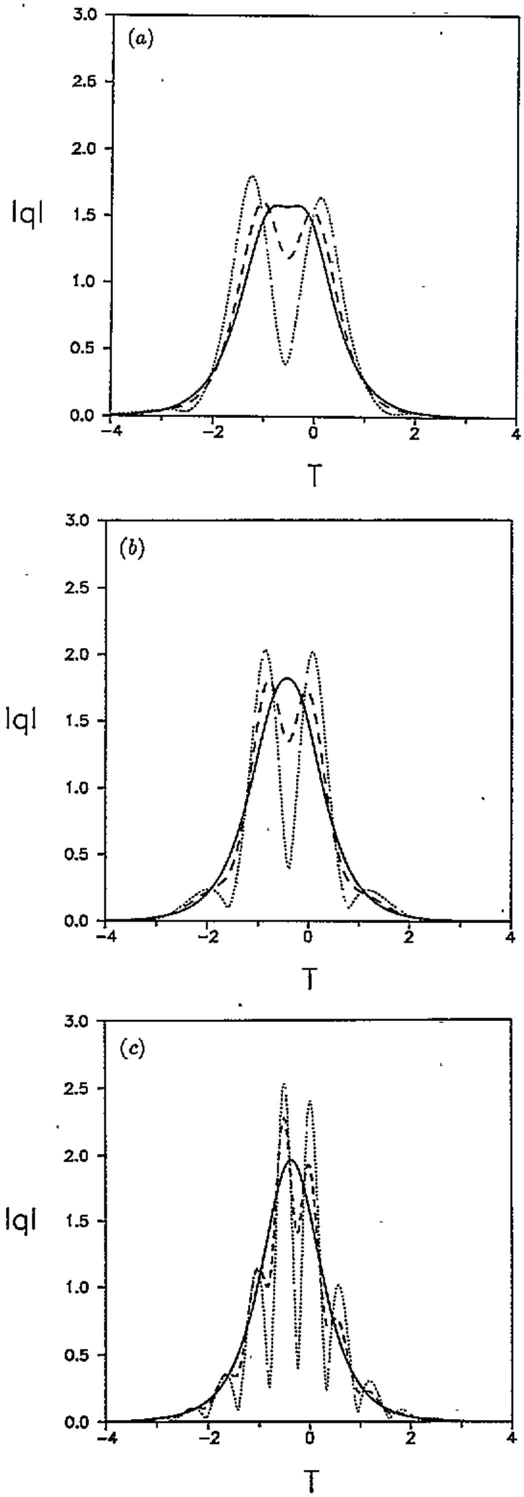

Figure 2. The shape of $|q|$ as a function of T for the general single soliton solution (26), with $\eta=2$, $|a_0|=1$, $\varphi_a=\varphi_b=0$. (a) $\xi=1$, (b) $\xi=2$, (c) $\xi=5$. Here $|b_0|=0$ (solid line), $|b_0|=0.25$ (dashed line) and $|b_0|=0.75$ (dotted line).

In order to obtain the 'breather' soliton solution one can choose $|a_0|=|b_0|=\kappa$ and $\arg a_0=-\arg b_0$. With this choice the soliton solution is:

$$u(x,t)=2\eta\,\frac{\kappa\,\mathrm{e}^{-A}[\eta\cos(B+2\phi)+4|\zeta|\sin(B+\phi)]-4\sqrt{2}|\zeta|^2\cos B\cosh(A+\psi)}{\kappa^2\,\mathrm{e}^{-2A}(1+\cos^2\phi)+\eta^2\cos[2(B+\phi)]-8|\zeta|^2\cosh^2(A+\psi)} \tag{27}$$

where $\psi=\ln\eta/\sqrt{2}\kappa$ and $\phi=\arg\zeta$.

This solution represents a pulse moving with the velocity $\varepsilon(\eta^2-3\xi^2)$ performing internal oscillations. Then the breather-like soliton solution of (1) in the case $\beta_1:\beta_2:\beta_3=1:6:3$ can be obtained from (27) via the transformation (2). In the limit $\xi=0$ the 'breather' soliton solution of (1) becomes the single-soliton solution (23) with T_0, φ_0 modified in view of our choice (25).

We notice that if one chooses $|a_0|=\kappa$, $|b_0|=0$ one can find the two-peak-soliton solution obtained in [26] (see equations (38)–(39) in [26]). In a similar way one can construct the N-soliton solution of (1) with $\beta_1:\beta_2:\beta_3=1:6:3$. The N-soliton solution written down in [26] was obtained only for a particular choice of $F(z)$, i.e., $c_{32}^{(i)*}=0$, $i=1,2,\ldots,N$ in (17).

In figure 1 we show the shape of the breather soliton solution (27) for the parameter values $\kappa=\sqrt{2}$, $\eta=2$, $\varphi_a=-\varphi_b=0$. Figure 2 shows the shapes of the general single soliton solution (26) for $\eta=2$, $\varphi_a=\varphi_b=0$, $|a_0|=1$, and for different values of ξ and $|b_0|$.

In spite of the fact that the ratio among the coefficients of the higher order terms $\beta_1:\beta_2:\beta_3$ in the PNLSE (1) is fixed at $1:6:3$, by an appropriate choice of the fibre parameters this situation can be realized in the femtosecond regime [6] so that we expect that the single soliton in the form (23) could be observed experimentally. A detailed analysis of the complete integrable PNLSE (1) for the case $\beta_1:\beta_2:\beta_3=1:6:3$ in the fibre optics context will be published elsewhere.

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